ANEC Problem set - Bootstrap School 2018

In the lectures, we discussed the relationships among: the ANEC, lightcone conformal blocks, and gravitational time delays. These are related to the behavior of the 4-point function

$$G(z,\bar{z}) = \frac{\langle \psi(0)\psi(z,\bar{z})O(1)O(\infty)\rangle}{\langle \psi(0)\psi(z,\bar{z})\rangle} \tag{0.1}$$

In these exercises, you will calculate the leading correction to this correlator four different ways, to see how they all agree. Set $d = 4$. Skip the methods you already are familiar with.

Method 1: Dolan-Osborn conformal block

The contribution of the identity and stress tensor block to this correlator give

$$G(z,\bar{z}) = 1 + \lambda_T g_{4,2}(z,\bar{z}) + \cdots \ . \tag{0.2}$$

with $\lambda_T = c_{\psi\psi T} c_{OOT}$. Assume this is the leading term in the lightcone limit, $\bar{z} \to 0$.

(i) Starting from the full Dolan-Osborn block $g_{4,2}(z,\bar{z})$, check that as $\bar{z} \to 0$,

$$g_{4,2}(z,\bar{z}) \approx \frac{\bar{z}z^3}{4} {}_2F_1(3,3,6,z) + O(\bar{z}^2) \tag{0.3}$$

This is the lightcone block.

(ii) The '2nd sheet' is defined by sending z around 1, ie $(1-z) \to (1-z)e^{-2\pi i}$. Show that on the second sheet, in the limit $\bar{z} \ll z \ll 1$, the leading term in the correlator (0.2) is

$$G(z,\bar{z}) \approx 1 + \lambda_T 90 i\pi \frac{\bar{z}}{z^2} \tag{0.4}$$

(iii) Set $\bar{z} = \eta\sigma$, $z = \sigma$, and show

$$\mathrm{Re} \int_S d\sigma (G(z=\sigma, \bar{z} = \eta\sigma) - 1) \propto \eta\lambda_T \tag{0.5}$$

The contour S is a semicircle in the lower half plane, centered at the origin. Find the coefficient.

This is the term which is positive, according to the sum rule derived in the lectures.

Method 2: Integrated Null Energy

The leading contribution of the identity plus stress tensor to the lightcone OPE is

$$\frac{\psi(u,v)\psi(-u,-v)}{\langle\psi(u,v)\psi(-u,-v)\rangle} \approx 1 - \frac{15c_{\psi\psi T}}{c_T} v u^2 \int_{-u}^{u} du' \left(1 - \frac{u'^2}{u^2}\right)^2 T_{uu}(u', v=0, x_\perp = 0) \quad (0.6)$$

Here u, v are Lorentzian lightcone coordinates: $u = t - y, v = t + y$.

(i) To check that this formula is correct, plug it into $\langle\psi\psi T\rangle$ and confirm that it gives the correct 3-point function in the lightcone limit. (This is one way to derive this formula.)

(ii) Translate into Euclidean $z, \bar{z}$, then plug this into (0.1) to derive (0.3).

(iii) For large u, (0.6) becomes

$$\frac{\psi(u,v)\psi(-u,-v)}{\langle\psi(u,v)\psi(-u,-v)\rangle} \approx 1 - \frac{15c_{\psi\psi T}}{c_T} v u^2 \int_{-\infty}^{\infty} du' T_{uu}(u', v=0, x_\perp = 0) \quad (0.7)$$

Plug this into (0.1) and show that it equals exactly the leading term, (0.4).

Note: In parts (ii) - (iii), you will need to a choose a contour for the u-integral to skirt the poles produced by the u insertion. This is a choice of 1st sheet vs 2nd sheet. What are the contour choices for the 1st and 2nd sheet? You should find that the integral in part (iii) is simply a residue from one of the O insertions.

Method 3: Geodesic length

Consider the asysmptotically AdS_5 spacetime

$$ds^2 = \frac{1}{z^2}(-dudv + d\vec{x}_\perp^2 + dz^2) + h_{ij}dx^i dx^j \quad (0.8)$$

I sketched in lecture how geodesics in this spacetime can be used to calculate lightcone OPEs. Here you'll work out the details.

(i) Set $h_{ij} = 0$ (ie the AdS vacuum). Find the geodesic with endpoints

$$(u_0, v_0, \vec{x}_\perp = 0) \quad \text{and} \quad (-u_0, -v_0, \vec{x}_\perp = 0) \quad (0.9)$$

Assume $v_0 < 0 < u_0$ so the geodesic is spacelike.

(ii) Now calculate δL, the change in this geodesic length from the first order correction in h_{ij}.

(iii) Take the limit $|v_0| \ll 1/u_0 \ll 1$.

(iv) Replace $h_{uu} = az^2 T_{uu}$ — this is the AdS/CFT dictionary in the limit $z \to 0$ — and show that δL reproduces (0.6) (for some choice of the constant a).