Bootstrap 2018 School: Integrability Problem Sets 3
Dispersion Relation from Matrix Quantum Mechanics

The goal of these exercises is to give some intuition about the physical origin of the dispersion relation of the magnons in $\mathcal{N}=4$ SYM at finite coupling. For this purpose, we will consider a certain matrix quantum mechanics which shares some properties with $\mathcal{N}=4$ SYM.

1 Dispersion relation and Zhukovsky variables

As explained in the lecture, the dispersion relation for the magnons describing the finite coupling $\mathcal{N}=4$ SYM is constrained by the symmetry[1] and is given by

$$e^{ip}=\frac{x^+}{x^-}\,,\qquad E=\frac{1}{2}\frac{1+\frac{1}{x^+x^-}}{1-\frac{1}{x^+x^-}}\,,\tag{1}$$

where $x^\pm$ are the so-called Zhukovsky variables

$$g\left(x(u)+\frac{1}{x(u)}\right)=u\,,\qquad x^\pm(u)\equiv x\left(u\pm\frac{i}{2}\right)\,,\tag{2}$$

with $g=\sqrt{\lambda}/(4\pi)$.

1. Expand the dispersion relation (1) at weak coupling ($g\ll 1$) and compare them with the dispersion relation of the SU(2) spin chain

$$e^{ip}=\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\,,\qquad E=\frac{g^2}{u^2+\frac{1}{4}}\,.\tag{3}$$

2. Show that the dispersion relation (1) is equivalent to

$$E=\sqrt{1+16g^2\sin^2\left(\frac{p}{2}\right)}\tag{4}$$

2. As a function of u, the energy and the momentum defined in (1) have two branch cuts; one associated with x^+ and the other associated with x^-. Check how they transform if you cross both branch cuts. (Physically this transformation corresponds to the crossing transformation.)

3. Show the following identity involving the Zhukovsky variables

$$\frac{x^+(u)-x^+(v)}{1-\frac{1}{x^-(u)x^-(v)}}=\frac{x^-(u)-x^-(v)}{1-\frac{1}{x^+(u)x^+(v)}}\,.\tag{5}$$

[1]Centrally extended $\mathfrak{psu}(2|2)$ symmetry.

2 Matrix quantum mechanics

One of the peculiar features of the dispersion relation shown above is the existence of the branch cuts in the u-plane. We now try to argue that these branch cuts come from the large N condensation of the eigenvalues of the matrices. Although there is some evidence supporting this interpretation[2], it is very hard to show this in $\mathcal{N} = 4$ SYM. Instead, here we analyze some simple matrix quantum mechanics[3], which shares certain properties with the $\mathcal{N} = 4$ SYM.

The model that we are going to study is given by the following action

$$S = \int dt \, \mathrm{tr} \left[\frac{1}{2}(\partial_t Z)(\partial_t \bar{Z}) + \frac{1}{2}(\partial_t X)(\partial_t \bar{X}) - \frac{1}{2}(Z\bar{Z} + X\bar{X}) - \frac{g_{\mathrm{YM}}^2}{8\pi^2}[Z, X][\bar{Z}, \bar{X}] \right] , \qquad (6)$$

where Z and X are $N \times N$ complex matrices and $\bar{Z}$ and $\bar{X}$ are their conjugates. This action can be obtained from $\mathcal{N} = 4$ SYM by putting the theory on $R \times S^3$, performing the dimensional reduction to R and keeping just a few terms neglecting all the other terms[4]. Furthermore, in the following analysis we assume that Z and $\bar{Z}$ are mutually commuting matrices, namely $[Z, \bar{Z}] = 0$. (We do not impose such conditions on other commutators.)

Before analyzing the model (6), let us consider an even simpler model

$$S = \int dt \, \mathrm{tr} \left[\frac{1}{2} \partial_t M \partial_t M - V(M) \right] , \qquad (7)$$

where M is a Hermitian $N \times N$ matrix. The Hamiltonian of this model is given by

$$H = \mathrm{tr} \left[-\frac{1}{2} \frac{\partial^2}{\partial M^2} + V(M) \right] , \qquad (8)$$

with

$$\mathrm{tr} \left[\frac{\partial^2}{\partial M^2} \right] = \sum_{a,b} \frac{\partial^2}{\partial M_{ab} \partial M_{ba}} \, . \qquad (9)$$

To analyze the model, it is useful to decompose the Hermitian matrix as

$$M = U^\dagger \Lambda \, U \, , \qquad (10)$$

where U is a unitary matrix and Λ is a diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \qquad (11)$$

[2]One can sometimes compute the same physical quantities both from localization and integrability. In such cases, by comparing the two computations, one finds that the branch cuts of the u-plane indeed correspond to the branch cuts formed by the condensation of the eigenvalues in the matrix model, which is obtained as a result of localization. See for instance [1, 2].

[3]As you will see, even in this simple matrix quantum mechanics, the argument is not completely rigorous. However it at least gives us some physical intuition into the origin of the branch cut.

[4]In [3], it was argued that the terms that we dropped are irrelevant for the quantity that we are computing. However, here we will refrain from discussing such points or making the connection to the original $\mathcal{N} = 4$ SYM, and just analyze the property of the model.

4. Show[5]

$$\frac{\partial \lambda_a}{\partial M_{ij}} = U_{ai} U^\dagger_{ja}\,,$$
$$\frac{\partial U^\dagger_{ia}}{\partial M_{jk}} = \sum_{b\neq a} \frac{U^\dagger_{ib} U_{bj} U^\dagger_{ka}}{\lambda_a - \lambda_b}\,, \qquad \frac{\partial U_{ia}}{\partial M_{jk}} = \sum_{b\neq a} \frac{U_{aj} U^\dagger_{kb} U_{bi}}{\lambda_a - \lambda_b}\,. \tag{12}$$

5. Using the result of the previous exercise, show

$$\mathrm{tr}\left[\frac{\partial^2}{\partial M^2}\right] = \frac{1}{\Delta(\lambda)} \sum_{a=1}^{N} \left(\frac{\partial}{\partial \lambda_a}\right)^2 \Delta(\lambda) + \sum_{a<b} \frac{K_{ab}}{(\lambda_a - \lambda_b)^2}\,, \tag{13}$$

with

$$\Delta(\lambda) \equiv \prod_{a<b} (\lambda_a - \lambda_b) \tag{14}$$

and K_{ab} is a differential operator made up of U, $U^\dagger$, $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial U^\dagger}$.

Since K_{ab} is an operator made up only of U's, it can be neglected as long as we talk about the singlet state, which is invariant under the $U(N)$ transformation

$$M \to g M g^\dagger \qquad \Longleftrightarrow \qquad \lambda_a : \text{fixed}\,, \quad U \to gU\,. \tag{15}$$

The restriction to the singlet states also simplify the analysis of the wave function (which is an energy eigenstate): Since the dependence on the U variables drop out, the wave functions for the singlet states are functions of only λ_a's. Therefore, it satisfies

$$\left[-\frac{1}{2} \frac{1}{\Delta(\lambda)} \sum_{a=1}^{N} \left(\frac{\partial}{\partial \lambda_a}\right)^2 \Delta(\lambda) + \sum_{a=1}^{N} V(\lambda_a)\right] \Psi(\lambda_1, \ldots, \lambda_N) = E \Psi(\lambda_1, \ldots, \lambda_N)\,, \tag{16}$$

Furthermore, due to the invariance under the $U(N)$ transformation, the wave function has to be symmetric with respect to the permutations of λ_a's. Finding an eigenstate is in particular simple if we consider

$$\Phi \equiv \Delta(\lambda) \Psi(\lambda_1, \ldots, \lambda_N)\,. \tag{17}$$

since the problem reduces to the standard multi-dimensional quantum mechanics,

$$\left[-\frac{1}{2} \sum_{a=1}^{N} \left(\frac{\partial}{\partial \lambda_a}\right)^2 + \sum_{a=1}^{N} V(\lambda_a)\right] \Phi = E \Phi\,. \tag{18}$$

Since the new wave function Φ is completely antisymmetric under the permutation of λ's, it can be thought of as the multi-particle wave functions of free fermions in a potential. More precisely, the ground state wave function Φ_{GS} is given by the Slater determinant,

$$\Phi_{\mathrm{GS}} = \det\left(\phi_n(\lambda_m)\right)\,, \tag{19}$$

[5] *Hint:* Use the relations $dM = d(U^\dagger \Lambda U)$, $U^\dagger = U^{-1}$ and $0 = d(U^\dagger U) = dU^\dagger U + U^\dagger dU$. (Also the identities like $\mathrm{tr}\,[\mathrm{diag}(1,0,0,\ldots)\cdot\Lambda] = \lambda_1$ might also be useful.)

where $\phi_n(\lambda)$ is a one-particle wave function of the n-th excited state

$$\left[-\frac{1}{2}\left(\frac{\partial}{\partial\lambda}\right)^2 + V(\lambda)\right]\phi_n = E_n\phi_n \,. \tag{20}$$

Let us now go back to our original problem. Since Z and $\bar{Z}$ are commuting, one can simultaneously diagonalize them as

$$Z \to UZU^\dagger = \mathrm{diag}(z_1, \ldots, z_N)\,, \qquad \bar{Z} \to U\bar{Z}U^\dagger = \mathrm{diag}(\bar{z}_1, \ldots, \bar{z}_N)\,. \tag{21}$$

The same transformation also maps X fields to $UXU^\dagger$, which we will again denote simply as X.

6. Show that the part of the Hamiltonian involving X fields can be written (after the diagonalization of Z) as

$$H_X = \sum_{a,b=1}^{N}\left[-\frac{\partial^2}{\partial X_{ab}\partial\bar{X}_{ba}} + \omega_{ab}^2 X_{ab}\bar{X}_{ba}\right]\,, \tag{22}$$

   with the frequency

$$\omega_{ab} \equiv \sqrt{1 + \frac{g_{\mathrm{YM}}^2}{2\pi^2}|z_a - z_b|^2}\,. \tag{23}$$

Disclaimer: The discussion so far has been rigorous. From now on, the argument becomes very handwaving. A more detailed discussion can be found in the paper [3] although even there not all the steps are completely justified.

Let us now analyze this Hamitlonian by treating z_a's as classical variables first and then later taking into account the path integral over z_a's. The state that we study is an analogue of the single-magnon state, which takes the following form

$$|p\rangle \sim \sum_n e^{ipn}\mathrm{tr}\left(\cdots ZZZ \underbrace{X}_{\text{n-th site}} ZZZ\cdots\right)|0\rangle_X \,. \tag{24}$$

Here $|0\rangle_X$ is the ground state of the X oscillators. After diagonalizing Z's we get

$$|p\rangle \sim \sum_{a,b}\sum_n e^{ipn}(z_a)^n \mathfrak{a}_{ab}^\dagger (z_b)^{L-n}|0\rangle_X = \sum_{a,b}\sum_n \left(e^{ip}\frac{z_a}{z_b}\right)^n z_b^L \mathfrak{a}_{ab}^\dagger|0\rangle_X \,. \tag{25}$$

Note that here we decomposed the X field X_{ab} into the creation and annihilation operator $X_{ab} = \mathfrak{a}_{ab}^\dagger + \mathfrak{a}_{ab}$ and use the fact that $\mathfrak{a}_{ab}$ annihilates the vacuum. As the wave function contains the factor

$$\sum_n \left(e^{ip}\frac{z_a}{z_b}\right)^n = \frac{1-\left(e^{ip}\frac{z_a}{z_b}\right)^{L+1}}{1-\left(e^{ip}\frac{z_a}{z_b}\right)}\,, \tag{26}$$

it is sharply peaked in the large L limit at

$$1 = e^{ip}\frac{z_a}{z_b}\,. \tag{27}$$

This means that, in the (path) integral over z_a's, we only need to consider the configurations which satisfy (27). The next step is to take into account of the integral over z_a's. It is however very hard to perform it rigorously in the presence of the interaction with the X fields. So let us just assume blindly that the ground state wave function for z_a's are exactly the same as the one for the free theory without any interaction with X's.

The ground state wave function for the matrix theory Z and $\bar{Z}$ can be obtained using the same technique as the ones described above. Here we will not discuss the detail of the computation (since anyway the whole discussion is hand-waving) and just state the final result

$$\Phi(z_1,\ldots,z_N) = \prod_{a<b}(z_a - z_b)\exp\left[-\sum_a |z_a|^2\right]\,. \tag{28}$$

The expectation value of z_a's can be computed from this wave function, and it turns out that in the large N limit, the dominant contribution is given by the region (which is basically the edge of the fermi surface of the quantum mechanics (20))

$$|z_a| = \sqrt{N/2}\,. \tag{29}$$

7. Evaluate the energy ω_{ab} (23) using the condition (27) and (29) and reproduce the dispersion relation of the $\mathcal{N}=4$ SYM spin chain (4).

8. Make the whole argument rigorous and write a paper!

References

[1] S. Giombi and S. Komatsu, "Exact Correlators on the Wilson Loop in $\mathcal{N}=4$ SYM: Localization, Defect CFT, and Integrability," JHEP **1805**, 109 (2018) arXiv:1802.05201.

[2] N. Gromov and G. Sizov, "Exact Slope and Interpolating Functions in N=6 Supersymmetric Chern-Simons Theory," Phys. Rev. Lett. **113**, no. 12, 121601 (2014) arXiv:1403.1894.

[3] D. Berenstein, D. H. Correa and S. E. Vazquez, "All loop BMN state energies from matrices," JHEP **0602**, 048 (2006) hep-th/0509015.