Pedro, problems.

1 Hexagons. Topology.

1. Consider a pair of pants, i.e. a sphere with three punctures to which we assign a weight g_{string}. Construct genus g surfaces with n punctures by multiplying together several pair of pants.

   Show that this naturally leads to a weight

$$g_{\text{string}}^{n+2g-2} . \tag{1}$$

2. Figure 1 represents an hexagonalization (or triangulation) of a four punctured sphere. Note that each external operator is split in this triangulation. Note that this decomposition of the Riemann surface is *not* a decomposition into pair of pants. In this picture each operator ($\equiv$ closed string) is split in three.

   Draw another hexagonalization where two operators are split in two and other two operators are split in four.

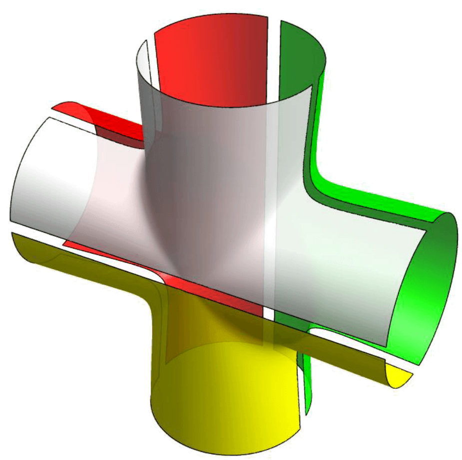

Figure 1: Hexagonalization of a 4pt correlation function $\equiv$ four closed string amplitude. (From the paper of Thiago Fleury and Shota Komatsu.)

3. Figure 1 has four hexagons. Draw similar figures for a three- and for a five- point functions. How many hexagons do they have? What do you expect for a planar n-point function? What about for an n-point function at genus g?

4. When we glue hexagons together we should sum over all possible states flowing through the lines along which the hexagons are glued. In figure 1 there are 6 such lines hence the four point function will be given by a six-fold sum. How many sums will we have for an n-point correlation function at genus g?

5. Figure 2 represents a (contribution to a) four point correlation function of local operators in a large N gauge theory at zero coupling where we simply Wick contract the constituents of each operator. Each line in the figure represents an arbitrary number of propagators connecting two operators. Hence each line is endowed with an integer number indicating how many propagators it contains. Such planar graphs (or ribbon graphs) with lengths associated to each line are called metric graphs. There are constraints on these integers of course since the number of propagators leaving a given operator is fixed to be the total number of fundamental fields of that composite operator.

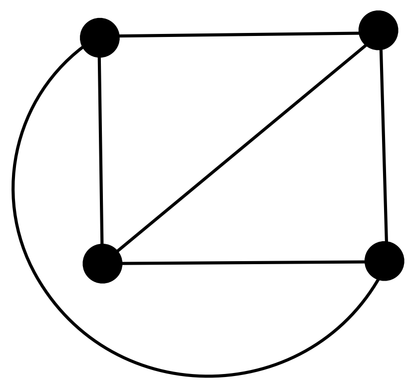

Figure 2: Free gauge theory four-point function.

   (a) Show that because of these constraints the sum over integers in the gauge theory becomes a two dimensional sum. This nicely matches the dimension of the moduli space of a four punctured sphere since there we integrate over the *complex* position of the fourth puncture once we gauge fix the position of the other three.

   (b) Note that figure 2 describes a triangulation of the four punctures sphere exactly like figure 1 does. Note that they are basically the same, there are again six lines dividing the four triangles/hexagons in this figure etc.

   What is the Feynman graph you would draw corresponding to the string picture in question 2 above.

   (c) Similarly, what Feynman diagrams would you draw in the gauge theory as the counterparts of the three- and five- string pictures of question 3 above.

   (d) Argue that the dimension of the discrete sums over lengths for a planar n-point function in the gauge theory nicely matches with the expected dimension of the string moduli space of a sphere with n punctures.

For further such connections check a beautiful paper by Shlomo Razzamat on the dual of the free gaussian matrix model.

2 Hexagons. Branch-Point-Twist-Field-Operators.

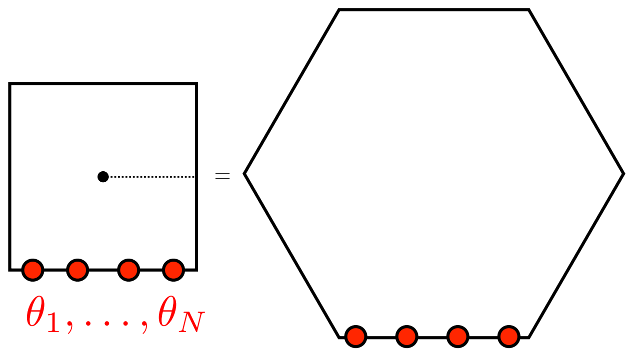

Figure 3: Hexagon twist field form factor for four particles.

The form factor of an M-gon branch-point twist field operators for a free massive boson can be represented in figure 3 reads

$$F_M = \langle 0|\Phi_M|\theta_1 \ldots \theta_n\rangle = \begin{cases} \sum\limits_{\text{pairs}} \prod\limits_{i,j} f_M(\theta_i - \theta_j) & , \quad n \text{ even} \\ \\ 0 & , \quad \text{n odd} \end{cases} . \tag{2}$$

where

$$f_M(\theta) = \frac{\sin(\frac{4\pi}{M})}{M/2 \sinh(\frac{i\pi+\theta}{M/2}) \sinh(\frac{i\pi-\theta}{M/2})} \tag{3}$$

1. Check that the form factor obeys the following equations:

$$\frac{1}{i}\text{Res}_{\theta_1 \to i\pi - \theta_2} F_M(\theta_1, \theta_2, \theta_3 \ldots, \theta_n) = F_M(\theta_3, \ldots, \theta_n) \tag{4}$$

$$F_M(\theta_1 + \frac{i\pi M}{2}, \theta_2 \ldots, \theta_n) = F_M(\theta_2, \ldots, \theta_n, \theta_1) \tag{5}$$

$$F_M(\theta_1, \theta_2, \theta_3 \ldots, \theta_n) = 1 \times F_M(\theta_2, \theta_1, \theta_3, \ldots, \theta_n) \tag{6}$$

and match then with the cartoons in figures 4a–4c.

2. Show that

$$\int_{\mathbb{R}+i\gamma} \frac{d\theta}{2\pi} e^{i\omega\theta} f_M(\theta) = \begin{cases} h_M(\omega)\,, & 2\pi/2 < \gamma < (M-2)\pi/2\,, \\ g_M(\omega)\,, & -2\pi/2 < \gamma < 2\pi/2\,, \\ k_M(\omega)\,, & -(M-2)\pi/2 < \gamma < -2\pi/2\,, \end{cases} \tag{7}$$

and compute g, h, k. Explain the geometrical meaning of these three branches. Hint: $g_M(\omega) = \sinh(\pi(n-1)\omega)/\sinh(\pi n\omega))$.

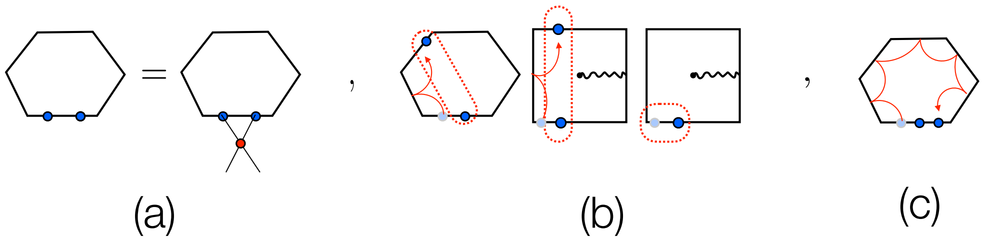

Figure 4: Watson Form Factors axioms for Branch Point Twist field Operators as proposed by J. L. Cardy, O. A. Castro-Alvaredo and B. Doyon,

3. Show that
$$\frac{h_{N_1} h_{N_2}}{1 - g_{N_1} g_{N_2}} \propto h_{N_1+N_2-4}\,, \tag{8}$$
and explain its physical meaning following the lecture.

4. Show that the two point function of branch-point twist field operators, computed by inserting a complete basis of states, nicely exponentiates as
$$\sum_{n \text{ even}} \frac{1}{n!} e^{-mx\sum_{j=1}^{n}\cosh(\theta_j)} |F_M(\theta_1 \ldots, \theta_m)|^2 = \exp\left(\sum_{n\text{even}} \frac{1}{2n} I_n\right) \tag{9}$$
with
$$I_n \equiv \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \ldots \int_{-\infty}^{\infty} \frac{d\theta_n}{2\pi} e^{-mx\sum_{j=1}^{n}\cosh(\theta_j)} f_M(\theta_1-\theta_2) f_M(\theta_2-\theta_3)\ldots f_M(\theta_n-\theta_1)$$

5. Show that for small mx we have
$$I_n \simeq \frac{\log(\frac{1}{mx})}{\pi} \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} \ldots \int_{-\infty}^{\infty} \frac{d\theta_n}{2\pi} f_M(0-\theta_2) f_M(\theta_2-\theta_3)\ldots f_M(\theta_n-0) \tag{10}$$
$$= 2\log(\frac{1}{mx}) \int \frac{d\omega}{2\pi} \left(g_M(\omega)\right)^n . \tag{11}$$

6. Combine the last two results to compare with the expected OPE of twist operators
$$\Phi_M \times \Phi_M = \Phi_{2M-4} + \ldots \tag{12}$$
which you should explain. You can use the dimension of the twist operators
$$\Delta(N) = \frac{c}{12}\left(\frac{4}{M} - \frac{M}{4}\right) \tag{13}$$
together with the identity
$$\int \frac{d\omega}{2\pi} \log(1 - (g_M(\omega))^2) = -\frac{(M-4)^2}{12M(M-2)}\,, \tag{14}$$
which you are also encouraged to prove.

7. Modify the last three points to analyse the OPE of $\Phi_{M_1} \times \Phi_{M_2}$ instead.

3 Hexagons. $\mathcal{N} = 4$ SYM.

1. Check the various hexagon axioms of the paper "Structure Constants and Integrable Bootstrap in Planar N = 4 SYM Theory" using the properties of the Zhukowsky variables, dispersion relation and Beisert S-matrix described in Shota's lectures.[1]

These hexagons are the building blocks of any n-point correlation function in this gauge theory at any genus. Note that the form factor sums were already *fun* to play with even for the free massive boson in the last problem so obviously life is not going to be trivial in $\mathcal{N} = 4$ and there is still a lot of beautiful structure to be unveiled which will for sure be of great use conceptually and pragmatically. Help is welcome!

[1] I am assuming people will not have time to get to this last problem since clearly i did not have time to prepare it :)