Problems on conformal kinematics

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`Mathematica` will be helpful in some of the exercises.

1 Embedding formalism

1. Show that in an n-point function ($n \leq d+2$) for fixed i there are only $n-2$ linearly independent $V_{i,jk}$. **Hint:** V's compute projection of Z_i onto the space spanned by X_i's. Relations $Z_i \cdot X_i = 0$ and $Z_i \sim Z_i + \lambda X_i$ remove some of the projections.

2. Show that H_{ij} and $V_{i,jk}$ suffice to write all parity-even tensor structures for n-point functions. **Hint:** use the approach from the first lecture by fixing n operators (in $\mathbb{R}^d$) to a $n-2$-dimensional subspace. Argue that V's describe projections of z on this subspace, and H describe inner products of components of z's orthogonal to the subspace. Explain why this suffices to describe parity-even invariants of stabilizer $\mathrm{SO}(d+2-n)$.

3. Consider a three-point function of 3 distinct spin-2 operators in 3d. Using our counting rules, show that there are 10 parity-even tensor structures. Then show that there there exist 10 different combinations of H_{ij} and $V_{i,jk}$. Note the symmetry properties

$$H_{ij} = H_{ji}, \quad V_{i,jk} = -V_{i,kj}. \tag{1}$$

   This means that there must exist a relation between these 10 combinations.

4. Check that the relation

$$(V_1 H_{23} + V_2 H_{13} + V_3 H_{12} + 2V_1 V_2 V_3)^2 + 2H_{12} H_{23} H_{13} = 0 \tag{2}$$

   holds at $X_i \cdot Z_i = X_i^2 = Z_i^2 = 0$ in 3d, where

$$V_1 = V_{1,23}, \quad V_2 = V_{2,31}, \quad V_3 = V_{3,12}. \tag{3}$$

   Hint: Notice that the six vectors Z_i, X_i all lie in a $\mathbb{R}^{d+1,1} = \mathbb{R}^{4,1}$ and thus must be linearly dependent. This implies that the 6x6 matrix of their inner products has a vanishing determinant.

5. Consider embedding formalism in 1d, with $X \in \mathbb{R}^{2,1}$.

   (a) Show that the constraint $X^2 = 0$ can be solved by

   $$X^m = \gamma^m_{\alpha\beta}\chi^a\chi^b, \tag{4}$$

   where χ is a spinor of $\mathrm{Spin}(2,1) \simeq \mathrm{SL}(2,\mathbb{R})$ and γ are gamma-matrices. We can thus describe primary operators as functions $\mathcal{O}(\chi)$.

   (b) Show that the action of the center of $\mathrm{Spin}(2,1)$ implies that

   $$\mathcal{O}(-\chi) = \pm\mathcal{O}(\chi). \tag{5}$$

   where the sign depends on whether $\mathcal{O}$ is bosonic or fermionic.

   (c) What is the two-point function of two fermions? How to project it back to physical space?