Problems on conformal kinematics

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Below is a bunch of mostly simple exercises (i.e. if you find yourself doing a hard calculation, you might be doing it wrong, unless the problem is marked as hard). One is encouraged to work on the problems which look more interesting to one's taste, although note that there are dependencies.

1 Tensor structures

Representation theory

1. (trivial) Check that only traceless-symmetric irreps reduce to trivial irrep under $\mathrm{Spin}(d) \to \mathrm{Spin}(d-1)$.

2. Let us say that two $\mathrm{Spin}(d)$ irreps belong to the same family iff they differ only by m_1 (i.e. the length of the first row in the Young diagram), and denote the family of ρ by $[\rho]$. Show that the families of $[\rho_d]$ of $\mathrm{Spin}(d)$ irreps which reduce (for representatives with sufficiently large m_1) to a given $\mathrm{Spin}(d-1)$ irrep ρ_{d-1} are in 1-to-1 correspondence with $\mathrm{Spin}(d-2)$ irreps to which ρ_{d-1} reduces,
$$[\rho_d] \ni \rho_{d-1} \ni \rho_{d-2}. \tag{1}$$

3. In the next two exercises we add a notion of parity to 2- and 3-dimensional Spin representations in order to talk more concretely about parity-odd and parity-even structures.[1] Let $\mathrm{Pin}_+(2)$ be the group consisting of elements (g, s), where $g \in \mathrm{Spin}(2)$ and $s = \pm 1$, with multiplication law
$$(g_1, s_1)(g_2, s_2) = (g_1 g_2^{s_1}, s_1 s_2). \tag{2}$$
In other words, $(1, -1)$ is a reflection and $(e^{i\phi/2}, 1)$ is a rotation by ϕ.
   (a) Show that irreps of $\mathrm{Pin}_+(2)$ consist of a scalar $\bullet^+$, pseudoscalar $\bullet^-$, and spin-j 2-dimensional irreps $\mathbf{j}$ for half-integer $j \geq \frac{1}{2}$.

[1] Note that this definition of parity may not be necessarily the one which is realized in a given CFT.

(b) Derive the multiplication rules

$$\begin{aligned}\bullet^{\pm} \otimes \mathbf{j} &= \mathbf{j}, \\ \mathbf{j}_1 \otimes \mathbf{j}_2 &= (\mathbf{j}_1 + \mathbf{j}_2) \oplus (\mathbf{j}_1 - \mathbf{j}_2), \qquad (j_1 \neq j_2), \\ \mathbf{j} \otimes \mathbf{j} &= (2\mathbf{j}) \oplus \bullet^{+} \oplus \bullet^{-}.\end{aligned} \tag{3}$$

(c) Show that under reduction to $O(1) = \mathbb{Z}_2$ integer-spin representations $\mathbf{j}$ reduce to $\bullet^{+} \oplus \bullet^{-}$, while $\bullet^{\pm}$ reduce to $\bullet^{\pm}$.

4. Let $\mathrm{Pin}_{+}(3)$ be the group generated by rotations $e^{i\sigma_i \phi/2}$ and reflections σ_i, where $i = 1, 2, 3$, and σ_i are the usual Pauli matrices. In other words, this is the group of unitary 2×2 matrices with determinant ± 1. The center is generated by

$$P = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \sigma_1 \sigma_2 \sigma_3. \tag{4}$$

(a) Explain how representations of $\mathrm{Spin}(3) \simeq \mathrm{SU}(2)$ naturally extend to $\mathrm{Pin}_{+}(3)$. Show that under this natural extension P has eigenvalue i^{2j} in spin-j representation. Check that this agrees with the intuitive action of P on tensor irreps. We will refer to these irreps as parity-even and denote them by j^{+}.

(b) Show that there is another family of irreps in which P has eigenvalue $-i^{2j}$. We will refer to these as parity-odd and denote them by j^{-}.

(c) Derive tensor product multiplication rules for $j^{\pm}$.

(d) Show that under reduction $\mathrm{Pin}_{+}(3) \to \mathrm{Pin}_{+}(2)$ we have

$$j^{\pm} \to \mathbf{j} \oplus (\mathbf{j} - 1) \oplus \cdots \oplus \mathbf{1} \oplus \bullet^{\pm}. \tag{5}$$

(Hint: A general hint to this exercise is to think about all irreps of Spin(3) as symmetric powers of the spinor irrep.)

Classification of correlation functions

1. Assume $\mathcal{O}$ is a primary operator. Show that if there is a differential operator $\mathcal{D}$ such that $\mathcal{D}\mathcal{O}$ is also a primary (for example, $\mathcal{O} = J$ is a current and $\mathcal{D}\mathcal{O} = \partial_\mu J^\mu$ is its divergence), then we necessarily have (at least at separated points)

$$\langle (\mathcal{D}\mathcal{O}) \mathcal{O}^\dagger \rangle = 0. \tag{6}$$

In other words, two-point functions automatically satisfy all conservation constraints, free field equations of motion, etc.

2. Using the character identities

$$\begin{aligned}
\mathrm{S}^2\chi(g) &= \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right), \\
\wedge^2\chi(g) &= \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right),
\end{aligned} \tag{7}$$

prove the character identity

$$S^2(\chi_1 - \chi_2) = S^2\chi_1 - \chi_1\chi_2 + \wedge^2\chi_2. \tag{8}$$

Using it, show that the counting rule that we derived for symmetric three-point tensor structures also works for conserved currents. (Hint: interpret the identity as number of structures minus number of conservation equations plus number of relations between these equations.)

3. Using our counting rules and representation theory exercises 3 and 4 above, recover the results of 1508.00012, 1705.04278, 1708.05718 for the operators $\mathcal{O}$ and the number of tensor structures appearing in (generic) 3d three-point functions

$$\langle\psi\psi\mathcal{O}\rangle, \quad \langle JJ\mathcal{O}\rangle, \quad \langle TT\mathcal{O}\rangle \tag{9}$$

respectively, taking parity into account. (Here J is abelian spin-1 current.)

4. How many four-point structures (or degrees of freedom) appear in (take parity into account)
   (a) $\langle\psi\psi\psi\psi\rangle$ in 3d
   (b) $\langle TTTT\rangle$ and $\langle JJJJ\rangle$ in 3d for abelian J. What about $\langle J_sJ_sJ_sJ_s\rangle$ (identical spin-s currents)?
   (c) $\langle JJJJ\rangle$ for non-abelian SU(2) flavor current J in 3d.
   (d) $\langle TTTT\rangle$ in 4d (somewhat annoying)

5. A conformal block participating in a given four-point function

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle \tag{10}$$

is labeled (among other things) by (ρ, a, b), where ρ is the representation of the exchanged operator $\mathcal{O}$ while a and b label the three-point tensor structures by which it couples to the external operators $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. In other words, a and b label the three-point tensor structures

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}^\dagger\rangle^{(a)} \text{ and } \langle\mathcal{O}_3\mathcal{O}_4\mathcal{O}\rangle^{(b)}. \tag{11}$$

For sufficiently large m_1 of ρ, the number of choices for a and b depends only on the family[2] $[\rho]$ (check this). Prove that the total number of choices of pairs of a and b over all families $[\rho]$ ("the number of conformal blocks") is the same as the number of four-point tensor structures

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle^{(c)}. \tag{12}$$

For simplicity, ignore permutation symmetries and parity, but take conservation into account. (Hint: use rep. theory exercise 2.)

[2] See the first rep. theory exercise.