Bootstrap Introduction

Generalized Free Field Structure Constants from Crossing

The problem

In this exercise, we will show that given a particularly simple spectrum of primaries appearing in the OPE, crossing symmetry allows us to uniquely fix the OPE coefficients.

Specifically, let ϕ be a scalar primary of dimension Δ_ϕ and assume that the $\phi \times \phi$ OPE contains the identity and for each $\ell = 0, 2, \ldots$ an infinite tower of operators of spin ℓ and dimension

$$\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell \,, \quad \text{where } n = 0, 1, \ldots \,. \tag{1}$$

We would like to determine expansion coefficients $a_{n,\ell} > 0$ so that the four-point function

$$\mathcal{G}(z,\bar{z}) = 1 + \sum_{n,\ell} a_{n,\ell} G_{\Delta_{n,\ell},\ell}(z,\bar{z}) \tag{2}$$

satisfies

$$\frac{\mathcal{G}(z,\bar{z})}{(z\bar{z})^{\Delta_\phi}} = \frac{\mathcal{G}(1-z,1-\bar{z})}{[(1-z)(1-\bar{z})]^{\Delta_\phi}} \,, \tag{3}$$

where $G_{\Delta,\ell}(z,\bar{z})$ is the conformal block of dimension Δ and spin ℓ. The unique solution of this problem is known as the generalized free field. Below, we will solve the problem in 2D for $\Delta_\phi = 1$.

1D warm-up

As a warm-up, let us solve the analogous problem in 1D CFTs in the special case $\Delta_\phi = 1/2$. The four-point function $\mathcal{G}(z)$ now depends on a single cross-ratio z and the crossing equation reads

$$\frac{\mathcal{G}(z)}{z^{2\Delta_\phi}} = \frac{\mathcal{G}(1-z)}{(1-z)^{2\Delta_\phi}} \tag{4}$$

Primary operators only carry the scaling dimension Δ (no spin), and the conformal blocks are

$$G_\Delta(z) = z^\Delta {}_2F_1(\Delta,\Delta;2\Delta;z) \,. \tag{5}$$

The 1D version of the spectrum (1) with $\Delta_\phi = 1/2$ is

$$\Delta_n = 1 + 2n \,, \quad \text{where } n = 0, 1, \ldots \,. \tag{6}$$

The task is to find a_n so that

$$z^{-1} + \sum_{n=0}^{\infty} a_n z^{-1} G_{2n+1}(z) = (1-z)^{-1} + \sum_{n=0}^{\infty} a_n (1-z)^{-1} G_{2n+1}(1-z) \,. \tag{7}$$

As explained in the lectures, this equation holds in the entire complex z plane away from the branch cuts $(-\infty, 0]$ and $[1, \infty)$.

a) To solve the problem, we will first study the branch cuts of the conformal blocks. Convince yourself that for generic Δ, $G_\Delta(z)$ has branch cuts stretching through $(-\infty, 0]$ and $[1, \infty)$, while for $\Delta \in \mathbb{N}$ only the second branch cut survives.

b) Using Mathematica or otherwise, convince yourself that the discontinuity of $G_\Delta(z)$ across the branch cut $[1,\infty)$ takes the form

$$\lim_{\epsilon\to 0^+}[G_\Delta(z+i\epsilon)-G_\Delta(z-i\epsilon)] = 2\pi i\,\frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2}p_\Delta\left(\tfrac{z-1}{z}\right)\,, \tag{8}$$

where

$$p_\Delta(x) = {}_2F_1(\Delta,1-\Delta;1;x); \tag{9}$$

Note that the change of variables $x=\frac{z-1}{z}$ maps the branch cut $z\in[1,\infty)$ to $x\in[0,1)$.

c) Show that for $\Delta\in\mathbb{N}$, $p_\Delta(x)$ is a polynomial. In fact, it is related to the Legendre polynomial $P_n(t)$ as follows[1]

$$p_\Delta(x) = (-1)^{\Delta-1}P_{\Delta-1}(2x-1)\,. \tag{10}$$

As a result, $p_\Delta(x)$ for $\Delta\in\mathbb{N}$ are orthogonal to each other

$$\int_0^1 p_j(x)p_k(x)\,dx = \frac{\delta_{jk}}{2j-1}\quad\text{for } j,k\in\mathbb{N}\,. \tag{11}$$

d) The above considerations make it possible to find a set of linear functionals dual to the set of conformal blocks $G_\Delta(z)$ with $\Delta\in\mathbb{N}$. Indeed, consider linear functionals ω_j whose action on a holomorphic function $\mathcal{F}(z)$ is defined as

$$\omega_j[\mathcal{F}] = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\, z^{-1}p_j\left(\tfrac{z-1}{z}\right)\mathcal{F}(z)\,. \tag{12}$$

$\mathcal{F}(z)$ should be holomorphic away from possible branch cuts $(-\infty,0]$ and $[1,\infty)$ and the integration contour passes in between the cuts.

By contour deformations and using the above results show that ω_j acts as follows on the objects entering the crossing equation (7)

$$\begin{aligned}
&\omega_j[z^{-1}] = 0\\
&\omega_j[z^{-1}G_k(z)] = \frac{\Gamma(2j-1)}{\Gamma(j)^2}\delta_{jk}\\
&\omega_j[(1-z)^{-1}] = 1\\
&\omega_j[(1-z)^{-1}G_k(1-z)] = 0\,.
\end{aligned} \tag{13}$$

for all $j,k\in\mathbb{N}$. Hint: Whenever the result is zero, there is a way to move the integration contour to infinity, where there is no contribution.

e) The natural next step would be to apply the functional ω_{2n+1} to the crossing equation (7) and observe that on the LHS, only the term with a_n survives, while on the RHS, only the identity contributes, allowing us to read off a_n. This would, however, lead to a wrong result for a_n! A simple way to see that something is amiss is to apply ω_{2n}, which kills all terms on the LHS, but equals 1 when acting on the RHS, giving a contradiction. Try to find the mistake in the argument before reading the explanation on the next page.

[1] This follows simply from the fact that the 1D conformal Casimir equation becomes the Legendre equation after a change of variables.

f) The hole in our reasoning appears because we allowed ourselves to interchange the action of the functional with the infinite sums over operators on either side of (7). This problem only occurs for functionals that depend on values arbitrarily close to the boundary of the crossing region (in our case the "chaos limit" $z = \infty$), and thus plays no role in the usual numerical bootstrap. You can read more about this subtle issue in [arXiv:1705.01357].

The claim (proven in the above paper), that we state here without proof, is that the action of a functional ω, defined using a holomorphic weight function $h(z)$ as

$$\omega[\mathcal{F}] = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} dz\, h(z)\mathcal{F}(z) \tag{14}$$

can be swapped with the infinite sum over operators if and only if $|h(z)| = O(|z|^{-1-\epsilon})$ as $z \to \pm i\infty$ with $\epsilon > 0$.

Show that for ω_j defined in (12), we have $h(z) \sim z^{-1}$ as $z \to \infty$, thus just missing the condition for swapping to be allowed.

g) There is now a simple way to correct our previous method: simply take linear combinations of ω_j that have the required $z \to \infty$ behaviour. Indeed, define the corrected functionals

$$\tilde{\omega}_j = \omega_j - (-1)^j \omega_2 \,. \tag{15}$$

Show that their weight functions satisfy $h(z) = O(z^{-2})$, thus satisfying the condition for swapping.

h) Show that applying $\tilde{\omega}_{2j}$ to the crossing equation (7) no longer leads to a contradiction and that applying $\tilde{\omega}_{2j+1}$ gives us the desired answer

$$a_n = \frac{2\Gamma(2n+1)^2}{\Gamma(4n+1)} \,. \tag{16}$$

You can verify that these are indeed the expansion coefficients of the 1D generalized free scalar four-point function for $\Delta_\phi = 1/2$

$$\mathcal{G}(z) = 1 + \frac{z}{1-z} + z \,. \tag{17}$$

2D at $\Delta_\phi = 1$

i) The method can be generalized to the same problem in 2D with $\Delta_\phi = 1$. This is because the 2D conformal blocks essentially factorize into 1D conformal blocks as follows

$$G^{2D}_{\Delta,\ell}(z,\bar{z}) = \frac{1}{1+\delta_{\ell,0}} \left[G^{1D}_{\frac{\Delta-\ell}{2}}(z) G^{1D}_{\frac{\Delta+\ell}{2}}(\bar{z}) + (z \leftrightarrow \bar{z}) \right] \,. \tag{18}$$

We now want to find $a_{n,\ell}$ so that

$$(z\bar{z})^{-1} + \sum_{\ell=0,2,\ldots} \sum_{n=0}^{\infty} a_{n,\ell} (z\bar{z})^{-1} G^{2D}_{2+2n+\ell,\ell}(z,\bar{z}) = (z \leftrightarrow 1-z, \bar{z} \leftrightarrow 1-\bar{z}) \,, \tag{19}$$

where $z, \bar{z}$ are independent complex variables and the equation holds whenever both z and $\bar{z}$ lie away from the branch cuts $(-\infty, 0]$ and $[1, \infty)$. Taking inspiration from the 1D case, we will consider functionals of the form

$$\omega[\mathcal{F}] = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\bar{z}}{2\pi i}\, h(z,\bar{z}) \mathcal{F}(z,\bar{z}) \,, \tag{20}$$

where $h(z, \bar{z})$ is holomorphic separately in z and $\bar{z}$ inside the crossing region. The condition for swapping such functionals with the sum over operators is

$$|h(z, \bar{z})| = O(|z|^{-1-\epsilon}|\bar{z}|^{-1-\epsilon}) \tag{21}$$

with $\epsilon > 0$ as $z, \bar{z} \to \pm i\infty$. We can now define a set of functionals $\omega_{j,k}$ by the weight functions

$$h_{j,k}(z, \bar{z}) = z^{-1} p_j\left(\tfrac{z-1}{z}\right) \bar{z}^{-1} p_k\left(\tfrac{\bar{z}-1}{\bar{z}}\right) \tag{22}$$

with $j, k \in \mathbb{N}$. Show that these functionals do not satisfy the swapping requirement.

j) By taking suitable linear combinations, define a set of functionals analogous to $\tilde{\omega}_j$ from (15) that "solves" the 2D crossing equation (19). [Hint: You may need to include not just $h_{j,k}(z, \bar{z})$ but also $h_{j,k}(z, 1 - \bar{z})$ in the basis.] You should find that the unique expansion coefficients take the form

$$a_{n,\ell} = 2\frac{\Gamma(n+1)^2\Gamma(n+\ell+1)^2}{\Gamma(2n+1)\Gamma(2n+2\ell+1)}\,, \tag{23}$$

corresponding to the four-point function

$$\mathcal{G}(z, \bar{z}) = 1 + \frac{z\bar{z}}{(1-z)(1-\bar{z})} + z\bar{z}\,. \tag{24}$$