Bootstrap School 2018 – Caltech

Exercises on Regge Physics

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1. Regge trajectories for the Coulomb potential

In this exercise, we will find the scattering amplitude of the non-relativistic particle in the Coulomb potential. We will then deduce the Regge poles from the scattering amplitude and compute the Regge trajectories.

Begin by writing the Schrodinger equation for this problem

$$\left[-\frac{1}{2m}\nabla^2 - \frac{U}{r}\right]\psi = E\,\psi \tag{1}$$

in terms of parabolic coordinates: $\xi = r + z$, $\eta = r - z$ and $\phi = \tan^{-1}\frac{y}{x}$

$$-\left[\frac{4}{\xi+\eta}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{4}{\xi+\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + \frac{1}{\xi\eta}\frac{\partial^2}{\partial\phi^2}\right]\psi - \frac{u}{\xi+\eta}\psi = k^2\psi \tag{2}$$

where $u = 4mU$ and $k = \sqrt{2mE}$ is the momentum of a free particle with mass m and energy E.
Choose the direction of incidence to be along the z-axis and appeal to axial symmetry to argue that the solutions must be independent of ϕ.
The solution we are looking for should have an incident plane wave part,

$$\exp(ikz) = \exp\left[i\frac{k}{2}(\xi-\eta)\right]\,,$$

and a scattered part which is radially outgoing,

$$\exp(ikr) = \exp\left[i\frac{k}{2}(\xi+\eta)\right]\,.$$

This suggests that we try the ansatz $\psi = \exp[\frac{ik}{2}(\xi-\eta)]g(\eta)$. Show that the resulting differential equation for $g(\eta)$ is the confluent hypergeometric equation:

$$\eta\frac{d^2g}{d\eta^2} + (1 - ik\eta)\frac{dg}{d\eta} + \gamma k g = 0$$

where $\gamma = \frac{u}{4k}$. Choosing the solution regular at the origin we arrive at:

$$\psi(\xi, \eta) = \exp\left[\frac{ik}{2}(\xi - \eta)\right] {}_1F_1(i\gamma; 1; ik\eta)$$

Use the large x behaviour of the hypergeometric function:

$${}_1F_1(a; b; x) \approx \frac{\Gamma(b)}{\Gamma(b-a)} \exp[-a \log(-x)] + \frac{\Gamma(b)}{\Gamma(a)} \exp[x + (a-b)\log(x)],$$

and write the wave function in the following form in the large r limit [1]

$$\psi \sim \exp\left[ikz - i\gamma \log k(r-z)\right] + \frac{f(k,\theta)}{r} \exp\left[ikr + i\gamma \log kr\right].$$

This shows how the Coulomb potential modifies the incident wave and the scattered wave even at large distances and leads to additional logarithmic phases. In analogy with the case of short range potentials, we define $f(k,\theta)$ to be the scattering amplitude. Show that

$$f(k,\theta) = \frac{\gamma}{k(1-\cos\theta)} \exp[i\gamma \log(1-\cos\theta) + 2i\sigma_0]$$

where σ_0 is the argument of $\Gamma(1-i\gamma)$ i.e $\Gamma(1-i\gamma) = |\Gamma(1-i\gamma)| \exp(i\sigma_0)$. Determine the differential and total cross section.

In order to calculate the Regge trajectories, we first expand the wave function in partial waves. Show that

$$\psi = \sum_{l=0}^{\infty} P_l(\cos\theta) w_l(r) \tag{3}$$

where

$$w_l(r) \sim \frac{e^{-ikr - i\gamma \log(2kr) + i\pi l}}{r} - \frac{e^{ikr + i\gamma \log(2kr)}}{r} \frac{\Gamma(l+1-i\gamma)}{\Gamma(l+1+i\gamma)}, \qquad r \to \infty. \tag{4}$$

Hint 1: In integrals of the form $\int dx(1-x)^{a-ib} P_l(x)$ it might be useful to use Rodrigues formula for the Legendre polynomial $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}[(x^2-1)^l]$ followed by repeated integration by parts.

[1] Assuming we don't look too close to the forward direction i.e $|r-z|$ is always large.

Hint 2: In integrals of the form $\int dx e^{ikrx} P_l(x) h(x)$ it might be useful to write $e^{ikrx} = \frac{1}{ikr}\frac{d}{dx}e^{ikrx}$ and integrate by parts to get the leading term at large r.

Conclude that

$$e^{2i\delta_l(k)} = \frac{\Gamma(l+1-i\gamma)}{\Gamma(l+1+i\gamma)}$$

and therefore the Regge trajectories are given by

$$l = \alpha_n(E) = -n - 1 + i\frac{mU}{\sqrt{2mE}}$$

where $n = 0, 1, 2, \ldots$. Plot the first 3 Regge trajectories in the (E, l) plane. Check that you reproduce the Hydrogen atom spectrum when you compute the energies of stable bound states.

2. Virasoro-Shapiro amplitude

The tree level 2 to 2 scattering amplitude of dilatons (massless scalars) in superstring theory is given by

$$\mathcal{T}(s,t) = 8\pi G_N \left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right) \frac{\Gamma\left(1-\frac{\alpha' s}{4}\right)\Gamma\left(1-\frac{\alpha' t}{4}\right)\Gamma\left(1-\frac{\alpha' u}{4}\right)}{\Gamma\left(1+\frac{\alpha' s}{4}\right)\Gamma\left(1+\frac{\alpha' t}{4}\right)\Gamma\left(1+\frac{\alpha' u}{4}\right)} \tag{5}$$

where G_N is the 10 dimensional Newton constant, α' is the square of the string length and s, t and u are the usual Mandelstam invariant satisfying $s + t + u = 0$.

The goal of this exercise is to use this explicit amplitude as a playground to test the methods of Regge theory. In particular, we can compute its high energy limit in two different ways. One way is to use Regge theory to determine the contribution of the leading Regge trajectory. Another way is just to use the asymptotic expansion of the Gamma function.

Start by using the Stirling approximation of the Gamma functions to derive the high energy behaviour

$$\mathcal{T}(s(1 \pm i\epsilon), t) \approx \frac{32\pi G_N}{\alpha'} e^{\mp \frac{i\pi\alpha' t}{4}} \frac{\Gamma\left(-\frac{\alpha' t}{4}\right)}{\Gamma\left(1+\frac{\alpha' t}{4}\right)} \left(\frac{\alpha' s}{4}\right)^{\alpha_0(t)}, \qquad s \to \infty, \tag{6}$$

where $\alpha_0(t) = 2 + \frac{1}{2}\alpha' t$.

Let us try to re-derive this result using Regge theory. First, we consider the t-channel partial wave expansion

$$\mathcal{T}(s,t) = \sum_{l=0}^{\infty} (2l+1) f_l(t) P_l(z)\,, \tag{7}$$

where $z = \cos\theta_t = 1 + \frac{2s}{t}$, P_l are the Legendre polynomials and the sum only includes even spins. Then we notice that the amplitude has poles at $t = 4n/\alpha'$ for $n = 0, 1, 2, \ldots$. Take the residue at these poles and derive

$$\sum_{l=0}^{\infty} (2l+1) P_l(z) \mathrm{Res}_{t=\frac{4n}{\alpha'}} f_l(t) = \mathrm{Res}_{t=\frac{4n}{\alpha'}} \mathcal{T}(s,t) = -\frac{128\pi G_N}{(\alpha' n!)^2} \left(\frac{nz}{2}\right)^{2n+2} + O(z^{2n})\,.$$

Notice that the sum over l truncates because this is a polynomial of z. Conclude that $f_l(t)$ has poles at $t = 4n/\alpha'$ for $n = l/2-1, l/2, l/2+1, l/2+2, \ldots$. Plot the particle spectrum in the (m^2, l) plane (Chew-Frautschi plot). Show that the first pole gives

$$f_l(t) \approx \frac{r(l)}{t - m^2(l)}\,, \qquad\qquad m^2(l) = \frac{2}{\alpha'}(l-2)\,, \tag{8}$$

and determine the residue $r(l)$. This pole can also be interpreted as the leading Regge trajectory

$$f_l(t) \approx \frac{\beta_0(t)}{l - \alpha_0(t)}\,, \qquad\qquad \alpha_0(t) = 2 + \frac{1}{2}\alpha' t\,. \tag{9}$$

Determine $\beta_0(t)$. Compare the contribution of the leading Regge trajectory to the scattering amplitude in the high energy limit with (6).

Extra: Extend the analysis to sub-leading terms in the large s expansion.