Analytic Bootstrap Methods: Exercise Set 1

1. A conformal block for a four-point function of identical real scalars is defined by

$$
\begin{aligned}
G_{\Delta,J}(z,\bar{z}) &= x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \langle 0|\phi(x_3)\phi(x_4)|\mathcal{O}|\phi(x_1)\phi(x_2)|0\rangle \\
|\mathcal{O}| &= \sum_{\alpha,\beta=\mathcal{O},P^\mu\mathcal{O},\ldots} |\alpha\rangle (\mathcal{N}^{-1})_{\alpha\beta} \langle\beta| \\
\mathcal{N}_{\alpha\beta} &= \langle\alpha|\beta\rangle
\end{aligned} \tag{1}
$$

Here $\mathcal{O}$ has dimension Δ and spin J, and α, β run over $\mathcal{O}$ and its descendants. The matrix elements are computed in radial quantization.

Using this expression, show that for small z,

$$
\begin{aligned}
G_{\Delta,J}(z,\bar{z}) &= z^{\frac{\Delta-J}{2}} k_{\Delta+J}(\bar{z}) + \ldots \qquad (z \ll 1) \\
k_{2\bar{h}}(\bar{z}) &= \bar{z}^{\bar{h}} {}_2F_1(\bar{h},\bar{h},2\bar{h},\bar{z}).
\end{aligned} \tag{2}
$$

Proceed as follows:

(a) Place the operators in a 2-dimensional plane at positions $x_1 = (0,0), x_2 = (z,\bar{z}), x_3 = (1,1), x_4 = \infty$, so that we have

$$
G(z,\bar{z}) = (z\bar{z})^{\Delta_\phi} \langle 0|\phi(1,1)\phi(\infty)|\mathcal{O}|\phi(z,\bar{z})\phi(0,0)|0\rangle \tag{3}
$$

The subgroup of the conformal group that preserves the 2-plane is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The generators for one of the $\mathrm{SL}(2,\mathbb{R})$'s are

$$
[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0. \tag{4}
$$

Similarly, the other $\mathrm{SL}(2,\mathbb{R})$ is generated by $\bar{L}_{-1,0,1}$. Finally, the remaining generators of the conformal group involve the transverse directions.
Using

$$
[L_0, \phi(z,\bar{z})] = \left(z\frac{\partial}{\partial z} + \frac{\Delta_\phi}{2} \right) \phi(z,\bar{z}), \tag{5}
$$

show that

$$
(z\bar{z})^{\Delta_\phi} \phi(z,\bar{z})\phi(0)|0\rangle = z^{L_0} \bar{z}^{\bar{L}_0} \phi(1,1)\phi(0,0)|0\rangle. \tag{6}
$$

(b) Using (6), argue that in the small z limit, the states $\alpha, \beta = \bar{L}_{-1}^n \mathcal{O}_{\bar{z}\cdots\bar{z}}$ $(n \geq 0)$ give the leading contribution to the block (1). Here $\mathcal{O}_{\bar{z}\cdots\bar{z}}$ has weight $\frac{\Delta-J}{2}$ with respect to L_0 and weight $\frac{\Delta+J}{2}$ with respect to $\bar{L}_0$.

(c) Evaluate the matrix element

$$\langle 0|\phi(\infty)\phi(1,1)|\alpha\rangle = \langle 0|\phi(\infty)\phi(1,1)\overline{L}_{-1}^{n}\mathcal{O}_{\overline{z}\cdots\overline{z}}|0\rangle \tag{7}$$

by taking derivatives of a conformally-invariant three-point function. Compute the matrix elements $\langle\alpha|z^{L_0}\overline{z}^{\overline{L}_0}\phi(1,1)\phi(0,0)|0\rangle$ in terms of the above result.

(d) Compute the norms $\langle\mathcal{O}_{\overline{z}\cdots\overline{z}}|\overline{L}_1^n\overline{L}_{-1}^n|\mathcal{O}_{\overline{z}\cdots\overline{z}}\rangle$ and perform the sum over n to derive (2).

2. This problem introduces some basic conformal integrals. All operators are scalars with principal series dimensions unless otherwise specified.

(a) The shadow transform of a scalar operator $\mathcal{O}$ is defined by

$$\mathbf{S}[\mathcal{O}](x) = \int d^d y \langle\widetilde{\mathcal{O}}(x)\widetilde{\mathcal{O}}(y)\rangle\mathcal{O}(y), \tag{8}$$

where $\langle\widetilde{\mathcal{O}}(x)\widetilde{\mathcal{O}}(y)\rangle$ is the standard conformally-invariant two-point structure for operators with dimension $\widetilde{\Delta} = d - \Delta$. By going to Fourier space, show that

$$\mathbf{S}^2 = \frac{\pi^d\Gamma(\frac{d}{2}-\Delta)\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)\Gamma(d-\Delta)} \equiv \mathcal{N}(\Delta,0). \tag{9}$$

That is, $\mathbf{S}^2$ is an integral kernel that is proportional to the identity transformation. More precisely, we have

$$\int d^d x_1 \frac{1}{x_{01}^{2\Delta}x_{12}^{2\widetilde{\Delta}}} = \mathcal{N}(\Delta,0)\delta(x_{02}). \tag{10}$$

(b) The star-triangle relation says that for scalar operators $\mathcal{O}_i$,

$$\begin{aligned}\langle\mathbf{S}[\mathcal{O}_1](x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle &= \int d^d x_1' \frac{1}{x_{11'}^{2\widetilde{\Delta}_1}}\frac{1}{x_{1'2}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31'}^{\Delta_3+\Delta_1-\Delta_2}} \\ &= S_{\Delta_1}^{\Delta_2,\Delta_3}\frac{1}{x_{12}^{\widetilde{\Delta}_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\widetilde{\Delta}_1}x_{31}^{\Delta_3+\widetilde{\Delta}_1-\Delta_2}},\end{aligned} \tag{11}$$

where

$$S_{\Delta_1}^{\Delta_2,\Delta_3} = \frac{\pi^{\frac{d}{2}}\Gamma(\Delta_1-\frac{d}{2})\Gamma(\frac{\widetilde{\Delta}_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\widetilde{\Delta}_1+\Delta_3-\Delta_2}{2})}{\Gamma(d-\Delta_1)\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2})}. \tag{12}$$

Check that this is consistent with (9).

(c) Let $\mathcal{O}, \mathcal{O}'$ be principal series scalars with dimensions $\Delta = \frac{d}{2} + is$, $\Delta' = \frac{d}{2} + is'$. Argue that conformal invariance fixes a "bubble integral" to have the form

$$\int d^d x_1 d^d x_2 \langle \mathcal{O}(x_0) \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle \langle \widetilde{\mathcal{O}}_1(x_1) \widetilde{\mathcal{O}}_2(x_2) \widetilde{\mathcal{O}}(x_3) \rangle$$
$$= A\delta(s - s')\delta(x_{03}) + B\delta(s + s') \frac{1}{x_{03}^{2\Delta}} \tag{13}$$

Here $\langle \cdots \rangle$ are the standard conformally-invariant three-point structures for the given representations. Compute the coefficients A and B as follows.

i. Take the shadow transform with respect to x_0 (or x_3) to derive a relation between A and B. It suffices to compute B.

ii. Use the star-triangle relation to perform the integral over x_2. The result is proportional to a conformal two-point integral

$$\int d^d x_1 \frac{1}{x_{01}^{\Delta+\Delta'} x_{13}^{2d-\Delta-\Delta'}}. \tag{14}$$

iii. Let $s + s' = t$. To compute the coefficient B, we would like to compute the non-local part of the above two-point integral (i.e. the part that is nonzero when x_{03} is nonzero). When t is nonzero, the integral is proportional to a delta-function, which is zero when x_{03} is nonzero. However, when t is zero, equation (10) is problematic because $\mathcal{N}(d/2, 0) = \infty$. To evaluate the two-point integral at nonzero x_{03}, let us think of it as a limit of a three-point integral

$$\lim_{\epsilon \to 0} \int d^d x_1 \frac{1}{x_{01}^{\Delta+\Delta'-\frac{i\epsilon}{2}} x_{13}^{2d-\Delta-\Delta'-\frac{i\epsilon}{2}} x_{41}^{i\epsilon}} \tag{15}$$

Evaluate this using the star-triangle formula, show that it is proportional to $\delta(t)$ and compute the coefficient.